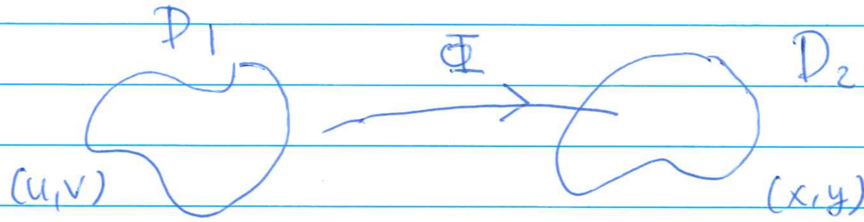
2020A
Fall 2020

Lecture 10

Chapter 3 Change of Variables.

- preparations.

Let Φ be a C^1-map 1-1 onto from $P_1 \to D_2$



$\Phi(u,v) = (\varphi_1(u,v), \varphi_2(u,v))$, φ_i are continuously differentiable

$= (x(u,v), y(u,v))$ (notations to simplify things, be cautious now x, y here are fcns, not variables!)

the Jacobian matrix of Φ is

$$J_\Phi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and its Jacobian determinant (or Jacobian) is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det J_\Phi$$
$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 1 Let $\Phi : D_1 \to D_2$ 1-1 onto C^1-map. Then

$$\iint_{D_2} F(x,y)\, dA(x,y) = \iint_{D_1} F(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v).$$

For conti F in D_2.

e.g. Consider $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$

$$J_\Phi = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \det J_\Phi = r$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$$

So, $\displaystyle\iint_{D_2} F\,dA = \iint_{D_1} F(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$ as before.

Idea behind the change of variables formula :

For simplicity take $D_1 = R$ a rectangle. Let P be a partition in R and under Φ, P introduces a partition (generalized) on D. Call R_j and D_j the subrectangles and sub-regions on R and D respectively. $D_j = \Phi(R_j)$. then

$$\iint_{D_2} F \sim \sum_j F(p_j)\,|D_j| \quad , \quad p_j \in D_j \text{ tag pt}$$

$$= \sum_j F(\Phi(q_j))\,\frac{|D_j|}{|R_j|}\,|R_j| \quad , \quad q_j \in R_j, \ \Phi(q_j) = p_j, \text{ possible as } \Phi \text{ 1-1, onto}$$

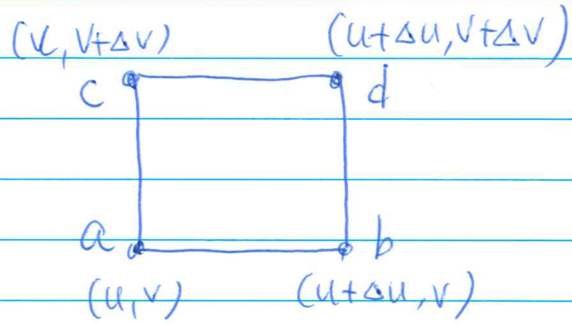
If we can show
$$\lim_{\|P\|\to 0} \frac{|D_j'|}{|R_j'|} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|, \qquad \text{(*)}$$

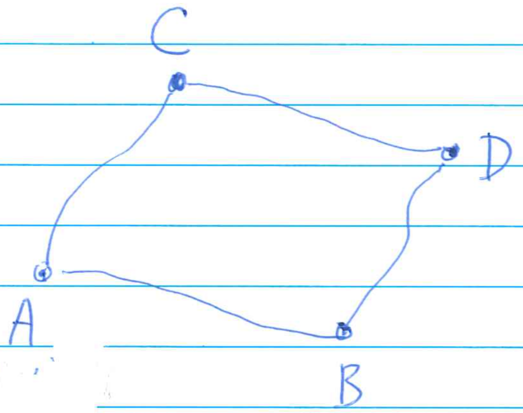
then

$$\iint_{D_2} F \to \iint_{D_1} F(\Phi(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v), \text{ as } \|P\|\to 0.$$

We are done.

Let's see how (*) is valid. Let $R = R_j$ and $D = D_j$.
$u = u_j$, $v = v_j$ etc





Under Φ, $a \to A = (x(u,v), y(u,v))$
$b \to B = (x(u+\Delta u, v), y(u+\Delta u, v))$
$c \to C = (x(u, v+\Delta v), y(u, v+\Delta v))$
$d \to D = (x(u+\Delta u, v+\Delta v), y(u+\Delta u, v+\Delta v))$

By Taylor's thm,

$$x(u+\Delta u, v) = x(u,v) + \frac{\partial x}{\partial u}(u,v)\,\Delta u + o(1)\,\Delta u$$
$$y(u+\Delta u, v) = x(u,v) + \frac{\partial y}{\partial u}(u,v)\,\Delta u + o(1)\,\Delta u$$

when $o(1) \to 0$ as $\Delta u \to 0$

$$x(u,v+\Delta v) = x(u,v) + \frac{\partial x}{\partial v}(u,v)\Delta v + o(1)\Delta v$$

$$y(u,v+\Delta v) = y(u,v) + \frac{\partial y}{\partial v}(u,v)\Delta v + o(1)\Delta v \quad , \ o(1) \to 0 \text{ as } \Delta v \to 0$$

$$x(u+\Delta u, v+\Delta v) = x(u,v) + \frac{\partial x}{\partial u}(u,v)\Delta u + \frac{\partial x}{\partial v}(u,v)\Delta v + o(1)\sqrt{(\Delta u)^2+(\Delta v)^2}$$

$$y(u+\Delta u, v+\Delta v) = y(u,v) + \frac{\partial y}{\partial u}(u,v)\Delta u + \frac{\partial y}{\partial v}(u,v)\Delta v + o(1)\sqrt{(\Delta u)^2+(\Delta v)^2}$$

$$, o(1) \to 0 \text{ a } \sqrt{(\Delta u)^2+(\Delta v)^2} \to 0$$

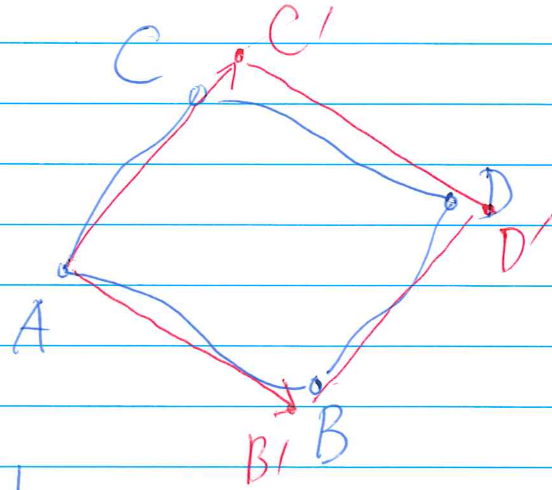
So B and $B' \equiv \left(x(u,v) + \frac{\partial x}{\partial u}(u,v)\Delta u,\ y(u,v) + \frac{\partial y}{\partial u}(u,v)\Delta u\right)$ are close up to $o(1)\Delta u$,

C and $C' \equiv \left(x(u,v) + \frac{\partial x}{\partial v}(u,v)\Delta v,\ y(u,v) + \frac{\partial y}{\partial v}(u,v)\Delta v\right)$ are close up to $o(1)\Delta v$,

D and $D' \equiv \left(x(u,v) + \frac{\partial x}{\partial u}(u,v)\Delta u + \frac{\partial x}{\partial v}(u,v)\Delta v,\right.$

$\left. y(u,v) + \frac{\partial y}{\partial u}(u,v)\Delta u + \frac{\partial y}{\partial v}(u,v)\Delta v\right)$

are close up to $o(1)\sqrt{(\Delta u)^2+(\Delta v)^2}$



$AB'C'D'$ form a parallelogram

whose area is

$$\left| \frac{\partial x}{\partial u}(u,v)\Delta u \frac{\partial y}{\partial v}(u,v)\Delta v - \frac{\partial x}{\partial v}(u,v)\Delta v \frac{\partial y}{\partial u}(u,v)\Delta u \right|$$

$$\therefore \lim_{\|P\|\to 0} \frac{|D_j|}{|R_j|} = \lim_{\|P\|\to 0} \frac{\left|\frac{\partial x}{\partial u}(u,v)\frac{\partial y}{\partial v}(u,v) - \frac{\partial x}{\partial v}(u,v)\frac{\partial y}{\partial u}(u,v) + o(1)\right|\Delta u \Delta v}{\Delta u \Delta v}$$

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| , \quad (*) \text{ holds.}$$

We'll present a more complete pf of Thm 1.

We need to recall some old facts.

Fact I (Inverse Function theorem) Let $\Phi : D \to \mathbb{R}^2$ be a C^1-map, $\Phi(u_0,v_0) = (x_0,y_0)$. If $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ at (u_0,v_0), the $\exists\, D_1$ containing (u_0,v_0), D_2 containing (x_0,y_0) s.t.

$\Phi\big|_{D_1} : D_1 \to D_2$ 1-1 onto so that its inverse is also C^1.

Fact II Consider $D_1 \xrightarrow{\Phi_1} D_2 \xrightarrow{\Phi_2} D_3$ both C^1-maps

then
$$J_{\Phi_2 \circ \Phi_1} = J_{\Phi_2} J_{\Phi_1}$$

and
$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(s,t)} .$$

Pf: $\Phi_1(s,t) = (u(s,t), v(s,t))$

$\Phi_2(x,y) = (x(u,v), y(u,v))$

$(\Phi_2 \circ \Phi_1)(s,t) = (x(u(s,t), v(s,t)), y(u(s,t), v(s,t)))$

Chain rule :

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s}, \quad \frac{\partial x}{\partial t} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t},$$

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial s}, \quad \frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

which is just

$$\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}, \text{ ie}$$

$$J_{\Phi_2 \circ \Phi_1} = J_{\Phi_2} J_{\Phi_1}.$$

~~Fact~~ Using $\det AB = \det A \det B$, we get

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(s,t)}.$$

Fact III. Suppose $\Phi : D_1 \to D_2$ is 1-1, onto, C^1. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}.$$

In particular, $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

Pf: $\Phi^{-1} \circ \Phi = Id$, ie $\Phi^{-1} \circ \Phi(u,v) = (u,v)$

So, $J_{\Phi^{-1}} J_{\Phi} = J_{id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore \det J_{\Phi^{-1}} \det J_{\Phi} = 1$, ie $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(u,v)} = 1$.